

Upper Bound For Matrix Operators On Some Sequence Spaces

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Intisari

Di dalam paper ini, akan didiskusikan masalah pencarian batas atas dan norma operator matrik Hausdorff pada beberapa ruang barisan.

Kata kunci: norma- F , fungsi- ϕ , matriks Hausdorff, batas atas.

Abstract

In this paper, we considered the problem of finding the upper bound and the norm of the Hausdorff matrix operator on some sequence spaces.

Keywords: F -norm, ϕ -function, Hausdorff matrix, upper bound.

1. Preliminaries and Some Basic Notions

Operator theory plays an important role in both pure and applied mathematics. Therefore, it always receives a lot of attention from mathematicians from those areas. In this paper, we discuss about the norm of a certain matrix operator on a certain sequence space. The key references are Jameson and Lashkaripour [2000], [2002], Lashkaripour [2002],[2004],[2005], and Pecari et.al [2001].

In this section, we give some basic notions. As usual, R and N denote the real and natural numbers system, respectively. R^+ denotes the collection of all positive real numbers. The collection of all sequences in R will be denoted by \mathcal{S} .

Let $X \subset \mathcal{S}$ be a linear space over R . A function $\| \cdot \| : X \rightarrow R$ is called an F -norm if it satisfies

- (i) $\|x\| \geq 0$ for every $x \in X$,
 $\|x\| = 0 \Leftrightarrow x = 0$,
- (ii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$, and
- (iii) if $\{x_n\} \subset X$ is a sequence such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ for some $x \in X$,
and $\{a_n\}$ is a sequence of real numbers which converges to some $a \in R$,
then $\lim_{n \rightarrow \infty} \|a_n x_n - ax\| = 0$.

The linear space X equipped with the F -norm $\|\cdot\|$, denoted $(X, \|\cdot\|)$, is called an F -normed space. When the F -norm $\|\cdot\|$ has been explicitly known, we write X instead of $(X, \|\cdot\|)$. An F -normed space is said to be complete if every Cauchy sequence in the space is convergent. A complete F -normed space is called a Fréchet space or shortly an F -space.

A function $\phi: R \rightarrow R$ is called a ϕ -function if it satisfies

- (i) $\phi(x) = 0 \Leftrightarrow x = 0$,
- (ii) $\phi(-x) = \phi(x)$, for every $x \in R$,
- (iii) ϕ is increasing on R^+ ,
- (iv) ϕ is continuous on R , and
- (v) $\lim_{x \rightarrow \infty} \phi(x) = \infty$.

A ϕ -function ϕ is said to satisfy a δ_2 -condition if there exists a real number $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \geq 0$. For any sequence of positive numbers $v = \{v_n\}$ and ϕ -function ϕ that satisfies δ_2 -condition, we define

$$l_\phi = \left\{ \{x_n\} \in \mathcal{S} : \sum_{n=1}^{\infty} \phi(x_n) < \infty \right\},$$

$$l_\phi(v) = \left\{ \{x_n\} \in \mathcal{S} : \sum_{n=1}^{\infty} v_n \cdot \phi(x_n) < \infty \right\}.$$

We observe that l_ϕ and $l_\phi(v)$ are complete F -norm spaces with respect to $\|\cdot\|_\phi$ and $\|\cdot\|_{\phi,v}$, respectively, where

$$\|x\|_\phi = \sum_{n=1}^{\infty} \phi(x_n) \quad \text{and} \quad \|x\|_{\phi,v} = \sum_{n=1}^{\infty} v_n \cdot \phi(x_n).$$

In case, $\phi(t) = |t|^p$, $1 \leq p < \infty$, we write $l_p(v)$ instead of $l_\phi(v)$.

Let $w = \{w_n\}$ be a decreasing positive sequence of real numbers such that

$\lim_{n \rightarrow \infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. We define

$$d(w, p) = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n (x_n^*)^p < \infty \right\}$$

where $\{x_n^*\}$ is a decreasing sequence which can be found by rearranging $\{|x_n|\}$. It can be shown that $d(w, p)$ is a space of all sequences with finitely non-zero elements. Further, $d(w, p)$ is an F -normed space with respect to

$$\|x\|_{d(w,p)} = \|x^*\|_{w,p}.$$

2. Matrix Operators

Let $\{a_n\}$ be a sequence of real numbers with $a_1 = 1$. For any $n \in \mathbb{N} \cup \{0\}$, we define the operator Δ^n as follows

$$\Delta^0 a_k = a_k, \Delta^1 a_k = a_k - a_{k+1}, \text{ and} \\ \Delta^n a_k = \Delta^{n-1}(\Delta^1 a_k), \quad n = 2, 3, 4, \dots$$

Further, the matrix $H = (h_{ij})$, where

$$h_{ij} = \begin{cases} C_{j-1}^{i-1} \cdot \Delta^{i-j} a_j, & 1 \leq j \leq i \\ 0, & j > i \end{cases}$$

is called the Hausdorff matrix.

Let μ be a probability measure on $[0, 1]$. For any $n \in \mathbb{N}$, we define the sequence $\{a_n\}$ by

$$a_n = \int_0^1 x^{n-1} d\mu(x), \quad n = 1, 2, 3, \dots$$

then we get the Hausdorff matrix $H(\mu) = (h_{ij})$, with

$$h_{ij} = \begin{cases} C_{j-1}^{i-1} \cdot \int_0^1 x^{j-1} (1-x)^{i-j} d\mu(x), & 1 \leq j \leq i \\ 0, & j > i \end{cases}$$

The followings are some kind of Hausdorff matrices:

1. $C(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \alpha(1-t)^{\alpha-1}$,
2. $H_0(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \frac{|\log t|^{\alpha-1}}{\Gamma(\alpha)} dt$, and
3. $G(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \alpha t^{\alpha-1} dt$,

where $\alpha > 0$ is any real number. The matrices $C(\alpha)$, $H_0(\alpha)$, and $G(\alpha)$ are called a Cesaro, Holder, and Gamma matrix respectively.

Let $v = \{v_n\}$ and $w = \{w_n\}$ be sequences of positive numbers. We consider the matrix operator $A: l_\phi(v) \rightarrow l_\phi(w)$

$$Ax = y = \{y_n\},$$

$$y_n = \sum_{j=1}^{\infty} a_{n,j} x_j.$$

The norm of A is given by

$$\|A\| = \sup \left\{ \|Ax\|_{\phi, w} : x \in l_{\phi}(v), \|x\|_{\phi, v} \leq 1 \right\}.$$

We observe the following theorem.

Theorem 2.1 Let $w = \{w_n\}$ be a decreasing sequence of positive real numbers. If the Hausdorff matrix operator $H(\mu)$ maps the space $l_{\phi}(w)$ into itself, then

$$\|Hx\|_{\phi, w} \leq \sup_{k \leq n} \frac{w_n}{w_k} \|x\|_{\phi, w}$$

Proof: For simplicity, we write $H(\mu) = H$. Take any $x \in l_{\phi}(w)$, then

$$\begin{aligned} \|Hx\|_{\phi, w} &= \sum_{i=1}^{\infty} w_i \cdot \phi \left(\sum_{j=1}^i C_{j-1}^{i-1} \cdot \left(\int_0^1 t^{j-1} (1-t)^{i-j} d\mu(t) \right) x_j \right) \\ &\leq \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} \phi(x_j) \\ &\leq \sup_{k \leq n} \frac{w_n}{w_k} \sum_{j=1}^{\infty} w_j \cdot \phi(x_j) = \sup_{k \leq n} \frac{w_n}{w_k} \cdot \|x\|_{\phi, w}. \end{aligned}$$

As a straight consequence, we then have the following corollary.

Corollary 2.2 If the Hausdorff matrix operator $H(\mu)$ maps the space l_{ϕ} into itself, then

$$\|H\|_{\phi, 1} \leq 1.$$

In case, the ϕ -function ϕ is of the form $\phi(x) = |x|^p$, $1 < p < \infty$, then we get inequalities for the Hausdorff matrix operator H .

Theorem 2.3 Let $v = \{v_n\}$ and $w = \{w_n\}$ be decreasing sequences of positive numbers, with $v_1 = 1$. If the Hausdorff matrix operator $H(\mu)$ maps $l_p(v)$ into $l_p(w)$, $1 < p < \infty$, then

$$\left(\inf \frac{w_n}{v_n} \right)^{1/p} \int_0^1 t^{-1/p} d\mu(t) \leq \|H\| \leq \left(\sup \frac{w_n}{v_n} \right)^{1/p} \int_0^1 t^{-1/p} d\mu(t).$$

Proof: We write $H(\mu) = H$ for the simplicity. Let $x \in l_p(v)$, then

$$\begin{aligned}
 \|Hx\|_{p,w}^p &= \sum_{i=1}^{\infty} w_i \cdot \left(\sum_{j=1}^i C_{j-1}^{i-1} \cdot \left(\int_0^1 t^{j-1} (1-t)^{i-j} d\mu(t) \right) x_j \right)^p \\
 &\leq \sum_{i=1}^{\infty} \sum_{j=1}^i w_j \cdot \left(C_{j-1}^{i-1} \cdot \left(\int_0^1 t^{j-1} (1-t)^{i-j} d\mu(t) \right) x_j \right)^p \\
 &\leq \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{i=1}^{\infty} \frac{w_i}{v_i} \cdot x_i^p \\
 &\leq \sup \frac{w_n}{v_n} \cdot \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{i=1}^{\infty} v_i \cdot x_i^p \\
 &= \sup \frac{w_n}{v_n} \cdot \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^p.
 \end{aligned}$$

These prove the right hand side of the inequality. Further, we are going to prove the left hand side of the inequality.

Let $0 < \delta < \frac{1}{p}$, $x_n = (n)^{-(1/p)-\delta}$, and $\varepsilon \in (0,1)$. It is clear that $\{x_n\} \in l_p$. Since

$0 < v_n \leq 1$ for every $n \in \mathbb{N}$, then $\{x_n\} \in l_p(v)$. Take α and N such that

$$\begin{aligned}
 \left(1 + \frac{1}{\alpha}\right)^{-2/p} &> 1 + \varepsilon, \\
 \int_{\alpha/n}^1 t^{-1/p} d\mu(t) &> (1 - \varepsilon) \int_0^1 t^{-1/p} d\mu(t), \quad n \geq N, \text{ and} \\
 \sum_{k=N}^{\infty} w_k x_k^p &> (1 - \varepsilon) \sum_{k=1}^{\infty} w_k x_k^p,
 \end{aligned}$$

then

$$\begin{aligned}
 (Hx)_n &= \sum_{k=1}^n C_{k-1}^{n-1} \left(\int_0^1 t^{k-1} (1-t)^{n-k} d\mu(t) \right) x_k \\
 &\geq (1 - \varepsilon)^2 x_n \cdot \int_0^1 t^{-1/p} d\mu(t), \quad n \geq N.
 \end{aligned}$$

Hence

$$w_n^{1/p} (Hx)_n \geq (1 - \varepsilon)^2 w_n^{1/p} x_n \cdot \int_0^1 t^{-1/p} d\mu(t), \quad n \geq N.$$

Further,

$$\begin{aligned}
\|Hx\|_{p,w}^p &= \sum_{n=N}^{\infty} w_n (Hx)_n^p \\
&\geq (1-\varepsilon)^{2p} \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} w_n x_n^p \\
&\geq (1-\varepsilon)^{2p+1} \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} w_n x_n^p \\
&\geq (1-\varepsilon)^{2p+1} \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} \frac{w_n}{v_n} v_n x_n^p \\
&\geq \inf \frac{w_n}{v_n} (1-\varepsilon)^{2p+1} \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^p.
\end{aligned}$$

These implies

$$\|Hx\|_{p,w}^p \geq \inf \frac{w_n}{v_n} \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^p.$$

If in the Theorem 2.4, we take $v_n = w_n$ for every n , then we get the following corollaries.

Corollary 2.5 If the Hausdorff matrix $H(\mu)$ maps the space $l_p(w)$ into itself, then

$$\|H\|_{p,w} = \int_0^1 t^{-1/p} d\mu(t).$$

Corollary 2.6 Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If the matrices $C(\alpha)$, $H_0(\alpha)$, and $G(\alpha)$ map the space $l_p(w)$ into itself, then

$$\|C(\alpha)\|_{w,p} = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}, \quad \alpha > 0$$

$$\|H_0(\alpha)\|_{w,p} = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{-1/p} |\log t|^{\alpha-1} dt, \quad \alpha > 0$$

$$\|G(\alpha)\|_{w,p} = \frac{\alpha p}{\alpha p - 1}, \quad \alpha p > 1.$$

Let $w = \{w_n\}$ be a monoton decreasing sequence of positive real numbers such that

$\lim_{n \rightarrow \infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. We define

$$d(w, p) = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n (x_n^*)^p < \infty \right\}$$

where $\{x_n^*\}$ is a monoton decreasing sequence found by rearranging the sequence $\{|x_n|\}$. It can be proved that $d(w, p)$ is a space that its members are all finite sequences. Further, $d(w, p)$ is an F -normed space with respect to

$$\|x\|_{d(w, p)} = \|x^*\|_{w, p}.$$

Lemma 2.7 Let $p \geq 1$ and $A = (a_{i,j})$ be the operator on $d(w, p)$ that satisfies

(i) $a_{i,j} \geq 0$ for every i, j , and

(ii) $\sum_{i \in M} \sum_{j \in K} a_{i,j} \leq \sum_{i=1}^m \sum_{j=1}^n a_{i,j}$ for every subset $M, K \subset \mathbb{N}$ that consists of m, n elements, respectively.

Then for every non negative elemen $x \in d(w, p)$, we have

$$\|Ax\|_{d(w, p)} \leq \|Ax^*\|_{d(w, p)}.$$

Proof: See Lashkaripour R. [2002].

Lemma 2.8 Let $p \geq 1$ and $A = (a_{ij})$ be an operator from $d(w, p)$ into itself such that $a_{ij} \geq 0$ for every i and j . If for every $x \in d(w, p)$,

$$Ax = \left(\sum_{j=1}^{\infty} a_{ij} x_j \right)^t$$

then the following statements are equivalent.

(a) $y_1 \geq y_2 \geq \dots \geq 0$ whenever $x_1 \geq x_2 \geq \dots \geq 0$.

(b) $r_{in} = \sum_{j=1}^n a_{ij}$ is a sequence such that $r_{(i+1)n} \leq r_{in}$ for every n .

Proof:

(a) \Rightarrow (b) : Let $x \in d(w, p)$ be an arbitrary, then $x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for some $n \in \mathbb{N}$. If $e_k = (0, \dots, 0, 1, 0, 0, \dots)$, that is a sequence with the k^{th} -coordinate is equal to 1

and the others are 0, then $x = \sum_{k=1}^n x_k e_k$. Further, by the hypothesis we have

$$0 \leq y_i - y_{i+1} = \sum_{j=1}^n (a_{ij} - a_{(i+1)j}) x_j.$$

(b) \Rightarrow (a) : If $x \in d(w, p)$, then $x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for some $n \in \mathbb{N}$. For any i , we have

$$\begin{aligned} y_i &= \sum_{j=1}^n a_{i,j} x_j = r_{i,1} x_1 + (r_{i,2} - r_{i,1}) x_2 + \dots + (r_{i,n} - r_{i,n-1}) x_n \\ &= r_{i,1} (x_2 - x_1) + r_{i,2} (x_3 - x_2) + \dots + r_{i,n} (x_n - x_{n-1}). \end{aligned}$$

Hence, $y_i \geq y_{i+1} \geq 0$ whenever $x_1 \geq x_2 \geq \dots \geq 0$.

Let $H(\mu)$ be a Hausdorff matrix such that $\sum_{i \in M} \sum_{j \in K} a_{ij} \leq \sum_{i=1}^m \sum_{j=1}^n a_{ij}$ for any subset $M, K \subset \mathbb{N}$, which consist of m, n elements, respectively. Following Lemma 2.7 and Lemma 2.8, then for any non negative decreasing sequence x we have

$$\|Hx\|_{d(w, p)} = \|Hx\|_{w, p}.$$

Further, by using Theorem 2.4, we have the following theorems.

Theorem 2.9 Let $p > 1$ and $H(\mu)$ be a Hausdorff matrix operator such that

$\sum_{i \in M} \sum_{j \in K} a_{ij} \leq \sum_{i=1}^m \sum_{j=1}^n a_{ij}$ for any subsets $M, K \subset \mathbb{N}$, which consist of m, n elements, respectively. Then $H(\mu)$ maps $d(w, p)$ into itself and

$$\|H\|_{d(w, p)} = \int_0^1 t^{-1/p} d\mu(t).$$

Theorem 2.10 Let $A = (a_{ij})$ be a matrix that satisfies the conditions (i) and (ii) in Lemma 2.7 and $\sum_{i=1}^{\infty} w_i a_{i1}$ be convergent. If $\{v_n\}$ is a sequence such that

$$\sup \frac{S_n}{V_n} < \infty$$

where $S_n = \sum_{k=1}^n s_k$, $s_n = \sum_{k=1}^{\infty} w_k \cdot a_{kn}$, and $V_n = \sum_{k=1}^n v_k$, then A is a bounded linear operator from $d(v,1)$ into $d(w,1)$ and

$$\|A\|_{v,w,1} = \sup \frac{S_n}{V_n}.$$

Proof: Let $x \in d(v,1)$ be sequence such that $x_1 \geq x_2 \geq \dots \geq 0$. If $M = \sup \frac{S_n}{V_n}$, then

$$\begin{aligned} \|Ax\|_{w,1} &= \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} s_j x_j \\ &= \sum_{j=1}^{\infty} S_j (x_j - x_{j+1}) \leq M \sum_{j=1}^{\infty} V_j (x_j - x_{j+1}). \end{aligned}$$

Since

$$\|x\|_{v,1} = \sum_{j=1}^{\infty} v_j x_j = \sum_{j=1}^{\infty} V_j (x_j - x_{j+1})$$

then

$$\|Ax\|_{w,1} \leq M \|x\|_{v,1}.$$

This implies $\|A\|_{v,w,1} \leq M$.

Further, by letting $x_1 = x_2 = \dots = x_n = 1$ and $x_{n+k} = 0$ for every $k \in \mathbb{N}$, then we have

$$\|x\|_{v,1} = V_n \text{ and } \|Ax\|_{w,1} = S_n.$$

So, $\|A\|_{v,w,1} = M$.

3. Concluding Remarks

In this paper, we have successfully constructed the sequence spaces $l_\phi(v)$ and $d(v, \phi)$, which is an F -space, respectively. Further, $d(v, \phi)$ is a sequence space where all of its elements are finite sequences. By restricting the function ϕ of the form $\phi(t) = |t|^p$, $1 \leq p < \infty$, then we can formulate the upper bound and norm of certain matrix operator on $l_p(v)$ and $d(v, p)$. The works will be continued for matrix operators act on $l_\phi(v)$ and $d(v, \phi)$.

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